Published for SISSA by Springer

Received: *April 14, 2016*
Accepted: *July 21, 2016*
Published: *August 2, 2016*

Boundary anomalies and correlation functions

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Abstract: It was shown recently that boundary terms of conformal anomalies recover the universal contribution to the entanglement entropy and also play an important role in the boundary monotonicity theorem of odd-dimensional quantum field theories. Motivated by these results, we investigate relationships between boundary anomalies and the stress tensor correlation functions in conformal field theories. In particular, we focus on how the conformal Ward identity and the renormalization group equation are modified by boundary central charges. Renormalized stress tensors induced by boundary Weyl invariants are also discussed, with examples in spherical and cylindrical geometries.

Keywords: Anomalies in Field and String Theories, Conformal and W Symmetry, Renormalization Group

ArXiv ePrint: 1604.02138

doi:10.1007/JHEP08(2016)013

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Contents

1 Introduction

As a non-trivial extension of Poincaré symmetry, Weyl invariance imposes significant constraints on the structure of correlation functions. For a generally non-conformal quantum field theory (QFT), the correlation functions approach those of a conformal field theory (CFT) sitting at one end point of the renormalization group (RG) flow at short distance, and those of a CFT sitting at another end point of the RG flow at long distance. It is therefore of great importance to search for a general principle constraining the flows between the two end points. In $d = 2$, the well-known c-theorem was proved in [1] thirty years ago. Conjectured first by [2], a proof of the a-theorem in $d = 4$ was given only recently by [3, 4], using the so-called dilaton anomaly effective action. The proof has not been found for $d = 6$ QFTs. See [5–8] for the recent progress. In proving these monotonicity theorems, the central charges, defined as the coefficients of the trace anomaly when embedding the theory in a curved background, play the central role. In particular, it is the central charge of the topological Euler density that satisfies the irreversibility of the RG flow.

There is no Weyl anomaly in odd-dimensional (compact) manifolds, and hence a definition for central charge becomes elusive. An alternative candidate that satisfies the

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monotonic behaviour along the RG flow in $d = 3$ was suggested by [9] as the Euclidean path integral of the CFT conformally mapped to S^3. See [10, 11] for further discussions on such an F-theorem; [12] points out an interpolation between the a-theorem and the F-theorem at the fixed points. The universal part of the partition function on S^3 can be further identified as the constant piece of the vacuum entanglement entropy (EE) across a disk. Moreover, with the help of strong subadditivity [13], the irreversibility of the RG flow in $d = 3$ can be proved [14]. It remains an important question whether one can link the fundamental properties of EE to the monotonicity theorem for spacetime dimensions higher than three. (See [15, 16] for examining the general RG flow using holography.)

However, in many physical systems, in particular in condensed matter physics, edge effects are important. It has also been shown recently that the universal part of EE can be understood purely as a boundary effect [17] (so in some sense the "area/boundary law" of EE is extended to also include the UV cut-off-independent log-term), which solves the main puzzle left from an earlier attempt [18]. The study of EE in $d = 3$ with a spacetime boundary is discussed in a more recent work [19]. Most important for this paper is the fact that there are additional boundary invariants in the presence of a boundary, for any-dimensional CFTs. To make a connection with the monotonicity theorem, we notice that the boundary central charge orders the boundary RG flow in $d = 3$ [20]; this b-theorem is then a generalization of the boundary g-theorem [21]. See [22] for a related discussion.

Motivated by these results, we here consider how the boundary central charges affect the conformal Ward identities and the RG equation of stress tensor correlation functions. We will largely focus on $d = 4$, but will also discuss $d = 3$ where only boundary anomalies exist. (In $d = 2$, there is no new boundary anomaly and only a boundary term of the Euler density is needed, which we discuss briefly in the appendix.)

To set the stage for our discussion, in the next section we first review the conformal invariance and the correlation function. The explicit expression for the stress tensor three-point function is rather bulky and we will refer the reader to [23, 24]. Our main focus here instead is the additional contribution to correlation functions from the local counter-terms (conformal anomaly), in particular when a manifold has a boundary. In section 3, we revisit several main identities of correlation functions. We are interested in correlation functions in flat bulk spacetime but we allow a generally curved, codimension-1 boundary. The boundary will be assumed to be compact and smooth (no corners). When revisiting these identities, we will not set the Weyl anomaly to be zero in the flat limit since the boundary terms survive even in the flat space. The boundary local counter-terms also contribute non-trivially to the stress tensor in the flat limit, so we will not drop the stress tensor either. These identities then generalize the ones given in the literature (for example, [23, 24]).

The $d = 4$ CFTs will be considered in section 4. We first discuss the RG equation of the three-point function and the conformal Ward identity for a compact manifold. Then, we generalize these results by introducing a boundary. It is found that the RG equation is modified by a special boundary central charge defined by a Weyl invariant constructed solely from the extrinsic curvature. The Ward identity is also modified due to the boundary counter-terms. Moreover, we obtain stress tensors in the vicinity of a boundary whose values are determined by boundary central charges. Examples in ball and cylindrical

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geometries are given. A similar analysis for $d = 3$ CFTs is in section 5, where the story becomes simpler because there is no bulk conformal anomaly and there are only two simple Weyl anomalies living on the boundary.

Among our new results, the most important ones are: a formal expression of the trace conformal Ward identity for general d (3.7); an RG equation for the three-point function (4.47), the Ward identity (4.48), a b_1-type stress tensor for a 4-cylinder (4.64), a-type stress tensors for a 4-ball (4.70) and for a 4-cylinder (4.73) for $d = 4$; the Ward identity (5.6), an RG equation for the two-point function (5.7), a c-type stress tensor for a 3-cylinder (5.13), a-type stress tensors for a 3-ball (5.17) and for a 3-cylinder (5.20) for $d = 3$.

In the conclusion we point out related questions. The appendix contains useful formulae for metric variation, and also a brief discussion on $d = 2$ CFTs with a boundary.

2 Conformal invariance and anomalous terms

For a conformal transformation g, $x_\mu \to x'_\mu = (gx)_\mu$, we can define a local orthogonal matrix on $\mathbb{R}^{d-1,1}$

$$\mathrm{R}^\mu{}_\nu(x) = \Omega^g(x)\frac{\partial x'^\mu}{\partial x^\nu}\,, \quad \mathrm{R}^\mu{}_\alpha(x)\mathrm{R}^\nu{}_\beta(x)\eta_{\mu\nu} = \eta_{\alpha\beta}\,, \tag{2.1}$$

where $\Omega^g(x)$ is the scale factor showing up in the transformed line element $ds'^2 = \Omega^g(x)^{-2}ds^2$. In the Euclidean signature we replace $\eta_{\mu\nu}$ by $\delta_{\mu\nu}$ on $\mathbb{R}^d$ and $\mathrm{R}_{\mu\nu}(x)$ becomes a local rotation matrix belonging to the group $O(d)$. One can generate a conformal transformation by a combination of translations, rotations and inversions, although the conformal group $\mathrm{SO}(d+1,1)$ can only be formed by even numbers of inversions since the inversion itself is not connected to the identity. If we consider an inversion through the origin, $x'_\mu = \frac{x_\mu}{x^2}$, we have $\Omega(x) = x^2$ and

$$\mathrm{R}_{\mu\nu}(x) = I_{\mu\nu}(x) = \delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}\,, \quad I^{\mu\nu}(x'-y') = \mathrm{R}^\mu{}_\alpha(x)\mathrm{R}^\nu{}_\beta(y)I^{\alpha\beta}(x-y)\,. \tag{2.2}$$

The matrix $I_{\mu\nu}(x-y)$ transforms like a vector and it can be regarded as a parallel transport for the conformal transformations. For three points, x, y, z, one can also define a covariant vector that transforms homogeneously. At the point z such a vector is given by

$$Z_\mu = \frac{1}{2}\partial^{(z)}_\mu \ln\frac{(z-y)^2}{(z-x)^2} = \frac{(x-z)_\mu}{(x-z)^2} - (x \leftrightarrow y)\,, \;\; Z'_\mu = \Omega(z)\mathrm{R}^\lambda{}_\mu(z)Z_\lambda\,. \tag{2.3}$$

Similarly, at points x, y one can define covariant vectors X_μ and Y_ν via cyclic permutation.

Denote a conformal primary operator as $\mathcal{O}^i(x)$ with i representing components in some representation of the rotation group. To construct the correlation functions, it is useful to adopt the induced representations [25] to write the conformal transformation as

$$\mathcal{O}^i(x) \to \mathcal{O}'^i(x') = \Omega(x)^\Delta D^i_j(\mathrm{R}(x))\mathcal{O}^j(x)\,, \tag{2.4}$$

where Δ is the conformal dimension; $D^i_j(\mathrm{R}(x))$ is the matrix in the associated representation. The conjugate representation of (2.4) is $\bar{\mathcal{O}}'_i(x') = \Omega(x)^\Delta\bar{\mathcal{O}}_j(x)(D(\mathrm{R}(x))^{-1})^j_i$, where

$\bar{\mathcal{O}}_i(x)$ stands for the conjugate field. The two-point function for conformal primary operators can be written by

$$\langle \mathcal{O}^i(x)\bar{\mathcal{O}}_j(y)\rangle = \frac{C_{\mathcal{O}}}{(x-y)^{2\Delta}} D^i_j(I(x-y))\,, \tag{2.5}$$

in the irreducible representations of $O(d)$. The overall constant $C_{\mathcal{O}}$ might be set to one by operator redefinition.

Our main subject of interest is the correlation functions of the stress tensor $T_{\mu\nu}$, which is a symmetric tensor satisfying $\partial^\mu T_{\mu\nu} = 0$ and $T^\mu_\mu = 0$. The latter property implies conformal invariance at the classical level. $T_{\mu\nu}$ has the scale dimension d required also by the conformal invariance. Now applying (2.5) to the stress tensor gives

$$\langle T_{\mu\nu}(x)T_{\alpha\beta}(y)\rangle = \frac{C_T}{(x-y)^{2d}}\mathcal{I}_{\mu\nu,\alpha\beta}(x-y)\,, \tag{2.6}$$

where C_T is a constant coefficient and

$$\mathcal{I}_{\mu\nu,\alpha\beta}(x-y) = I_{\mu\lambda}(x-y)I_{\nu\rho}(x-y)L^{\lambda\rho}{}_{\alpha\beta}\,, \quad L^{\lambda\rho}{}_{\alpha\beta} \equiv \frac{1}{2}\left(\delta^\lambda_\alpha\delta^\rho_\beta + \delta^\rho_\alpha\delta^\lambda_\beta - \frac{2}{d}\delta_{\alpha\beta}\delta^{\lambda\rho}\right). \tag{2.7}$$

Here $L^{\lambda\rho}{}_{\alpha\beta}$ is a projection operator onto the space of a symmetric, traceless tensor. The expression of the three-point function becomes rather involved [23, 24]. To keep the expression simple we will not list its full expression but simply write it as

$$\langle T_{\mu\nu}(x)T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle = \Gamma_{\mu\nu\sigma\rho\alpha\beta}(x,y,z)\,. \tag{2.8}$$

The structure of $\Gamma_{\mu\nu\alpha\beta\sigma\rho}(x,y,z)$ is determined by the conformal symmetry and the stress tensor conservation. Its explicit form will not be too relevant in the present discussion. However, it is important to know that there are three independent forms (in $d=3$ there are two and only one in $d=2$) for the stress tensor three-point function [23].

Here we focus on the additional contribution to (2.8) coming from possible local counter-terms in the effective action. These counter-terms give precisely the Weyl anomaly. We will adopt dimensional regularization working in $d \to d+\epsilon$ dimensions and refer to these counter-terms as the anomaly effective action denoted by $\widetilde{W} \equiv \frac{\mu^\epsilon}{\epsilon}\tilde{W}$.[1] By including possible counter-terms we have

$$\langle T_{\mu\nu}(x)T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle = \Gamma_{\mu\nu\sigma\rho\alpha\beta}(x,y,z) + (-2)^3\frac{\delta^3}{\delta g^{\mu\nu}(x)g^{\sigma\rho}(y)\delta g^{\alpha\beta}(z)}\widetilde{W}\,. \tag{2.9}$$

In $d=4$, there are two types of local counter-terms: one type-A anomaly and one type-B anomaly. (We will remove the scheme-dependent type-D total derivative anomaly.) The contributions from the bulk anomalies were first calculated by [23] and we will review them. Notice that the scale μ-dependent part in the second term of (2.9) determines the RG equation of the three-point function.

[1] $\widetilde{W}$ represents the counterterms (the terms proportional to the anomaly) that must be added to regularize divergences when placing the CFT in a curved spacetime.

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Our main input is the new correction from the boundary terms of the effective action, i.e. the second term in

$$\widetilde{W} = \frac{\mu^\epsilon}{\epsilon}\Big(\int_{\mathcal{M}} \tilde{W} + \int_{\partial\mathcal{M}} \tilde{W}_{\rm bry}\Big)\,. \tag{2.10}$$

The effective action must reproduce the boundary anomalies via the conformal transformation. As we will discuss in more detail, in addition to the Euler characteristic boundary term needed to preserve topological invariance, there exists new boundary Weyl invariants. To our knowledge, the corrections to the stress tensor correlation functions from the boundary anomalies have not been discussed in the literature.

3 Ward identities and anomalies revisited

Considering a field theory coupled to a non-dynamical, curved background, we define the stress tensor in the Euclidean signature by

$$\langle T_{\mu\nu}(x)\rangle = -\frac{2}{\sqrt{g}}\frac{\delta W}{\delta g^{\mu\nu}(x)}\,, \tag{3.1}$$

where the functional $W \equiv W(g(x))$ is the effective action (the generating functional for connected Green's functions). The classical stress tensor trace vanishes for CFTs but at the quantum level, the regularization and renormalization introduce conformal symmetry breaking counter-terms that result in $\langle T^\mu_\mu(x)\rangle \neq 0$; $\widetilde{W}$ denoted earlier is a part of W. We assume the theory is regulated in a diffeomorphism-invariant way and we will focus on the vacuum. The trace anomaly is a function of intrinsic curvatures (for a compact manifold) that must vanish in the flat limit. However, the flat space correlation functions depend on central charges: a result reminding us to adopt the correct order of limits, that the flat limit is imposed only after performing the metric variation.

We have the following definitions of the functional differentiation:

$$\langle T_{\mu\nu}(x)T_{\sigma\rho}(y)\rangle = \lim_{g_{\mu\nu}\to\delta_{\mu\nu}}\Big((-2)^2\frac{\delta^2}{\delta g^{\sigma\rho}(y)\delta g^{\mu\nu}(x)}W|_{g=0}\Big)\,, \tag{3.2}$$

$$\langle T_{\mu\nu}(x)T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle = \lim_{g_{\mu\nu}\to\delta_{\mu\nu}}\Big((-2)^3\frac{\delta^3}{\delta g^{\alpha\beta}(z)\delta g^{\sigma\rho}(y)\delta g^{\mu\nu}(x)}W|_{g=0}\Big)\,, \tag{3.3}$$

where $W|_{g=0}$ is the standard procedure getting rid of the source term in the action after performing the variation. Notice that we have restricted the results to flat space at the end of the computation. From these definitions, we can derive the following identity:

$$\begin{aligned}&\langle T_{\mu\nu}(x)T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle\\ &= -8\lim_{g_{\mu\nu}\to\delta_{\mu\nu}}\frac{\delta}{\delta g^{\alpha\beta}(z)}\frac{1}{\sqrt{g(y)}}\frac{\delta}{\delta g^{\sigma\rho}(y)}\frac{1}{\sqrt{g(x)}}\frac{\delta}{\delta g^{\mu\nu}(x)}W|_{g=0} + B_{\mu\nu\sigma\rho\alpha\beta}(x,y,z)\\ &\quad+\delta_{\alpha\beta}\Big(\delta^d(y-z)+\delta^d(x-z)\Big)\langle T_{\mu\nu}(x)T_{\sigma\rho}(y)\rangle + \delta_{\sigma\rho}\delta^d(x-y)\langle T_{\mu\nu}(x)T_{\alpha\beta}(z)\rangle\,,\end{aligned} \tag{3.4}$$

where

$$B_{\mu\nu\sigma\rho\alpha\beta}(x,y,z) = \Big(\delta_{\sigma\alpha}\delta_{\rho\beta}+\delta_{\sigma\beta}\delta_{\rho\alpha}-2\delta_{\alpha\beta}\delta_{\sigma\rho}\Big)\delta^d(x-y)\delta^d(y-z)\langle T_{\mu\nu}(x)\rangle|_{\delta_{\mu\nu}} \tag{3.5}$$

vanishes for a manifold without a boundary because the expectation value of the stress tensor vanishes in flat space. However, we will find a non-trivial contribution to $\langle T_{\mu\nu}(x)\rangle|_{\delta_{\mu\nu}}$ when we include boundary Weyl anomalies. The conservation of the stress tensor implies

$$\begin{aligned} &\partial^\mu\langle T_{\mu\nu}(x)T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle \\ &=\partial_\nu\Big(\delta^d(x-y)+\delta^d(x-z)\Big)\langle T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle+\partial^\mu B_{\mu\nu\sigma\rho\alpha\beta}(x,y,z) \\ &\quad+\Big\{\partial_\sigma\Big(\delta^d(x-y)\langle T_{\rho\nu}(x)T_{\alpha\beta}(z)\rangle\Big)+\sigma\leftrightarrow\rho\Big\}+\Big\{\partial_\alpha\Big(\delta^d(x-z)\langle T_{\nu\beta}(x)T_{\sigma\rho}(y)\rangle\Big)+\alpha\leftrightarrow\beta\Big\}. \end{aligned} \tag{3.6}$$

An important relation that we can also derive from these definitions is the following trace conformal Ward identity relating the three-point function and the two-point function:

$$\begin{aligned} &\langle T^\mu_\mu(x)T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle \\ &=2\Big(\delta^d(x-y)+\delta^d(z-x)\Big)\langle T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle+4\lim_{g_{\mu\nu}\to\delta_{\mu\nu}}\Big(\frac{\delta^2}{\delta g^{\alpha\beta}(z)\delta g^{\sigma\rho}(y)}\langle T^\mu_\mu(x)\rangle\Big) \\ &\quad-2\lim_{g_{\mu\nu}\to\delta_{\mu\nu}}\Big(\delta_{\alpha\beta}\delta^d(x-z)\frac{\delta}{\delta g^{\sigma\rho}(y)}+\delta_{\sigma\rho}\delta^d(x-y)\frac{\delta}{\delta g^{\alpha\beta}(z)}\Big)\langle T^\mu_\mu(x)\rangle \\ &\quad+S_{\alpha\beta\sigma\rho}(x,y,z)\langle T^\mu_\mu(x)\rangle|_{\delta_{\mu\nu}}\,, \end{aligned} \tag{3.7}$$

where

$$S_{\alpha\beta\sigma\rho}(x,y,z)=\Big(\delta_{\alpha\sigma}\delta_{\beta\rho}+\delta_{\alpha\rho}\delta_{\beta\sigma}+\delta_{\alpha\beta}\delta_{\rho\sigma}\Big)\delta^d(x-y)\delta^d(x-z)\,. \tag{3.8}$$

For CFTs in a compact manifold the last two lines of (3.7) do not contribute, and therefore these terms are not included in the literature. (Take $d=4$ CFTs for example. After performing a metric variation on the stress tensor trace, the result vanishes in the flat limit since the trace anomaly is a function of curvature squared. The same argument applies for higher dimensions.) We see that in general the existence of the boundary Weyl anomaly modifies the trace Ward identity.

4 Four-dimensional CFTs

4.1 Compact manifold

In this section we reproduce several main results for $d=4$ CFTs without boundary terms. We follow [23, 24] closely but the way we adopt the dimensional regularization will be slightly different in the details.

The conformal anomaly reads

$$\langle T^\mu_\mu(x)\rangle=\frac{1}{16\pi^2}\Big(cW^2_{\mu\sigma\rho\nu}-aE_4+\gamma\Box R\Big)\,. \tag{4.1}$$

The type-A anomaly is defined by the Euler density,

$$E_4=\frac{1}{4}\delta^{\alpha\beta\gamma\delta}_{\mu\nu\sigma\rho}R^{\mu\nu}{}_{\alpha\beta}R^{\sigma\rho}{}_{\gamma\delta}=R^2_{\mu\nu\sigma\rho}-4R^2_{\mu\nu}+R^2\,, \tag{4.2}$$

where $\delta^{\alpha\beta\gamma\delta}_{\mu\nu\sigma\rho}$ is the fully anti-symmetrized product of four Kronecker delta functions. The Euler number χ is given by $\int_{\mathcal{M}}d^4x\sqrt{g}E_4=4\pi\chi$. The type-B anomaly defines the central

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charge c; the Weyl tensor in d-dimensions (for $d > 3$) is given by

$$W^{(d)}_{\mu\sigma\rho\nu} = R_{\mu\sigma\rho\nu} - \frac{2}{d-2}\left(g_{\mu[\rho}R_{\nu]\sigma} - g_{\sigma[\rho}R_{\nu]\mu} - \frac{g_{\mu[\rho}g_{\nu]\sigma}}{(d-1)}R\right), \tag{4.3}$$

which has basic properties: $W_{\mu\sigma\rho\nu} = W_{[\mu\sigma][\rho\nu]}$, $W_{\mu[\sigma\rho\nu]} = 0$ and $W^{\mu}{}_{\sigma\rho\mu} = 0$. The last term in (4.1), referred to as the type-D anomaly, is scheme-dependent: adding an R^2 local counter-term shifts its anomaly coefficient. Here we use this freedom to remove this anomaly. See [26] and [27] for related discussions on the type-D anomaly.[2]

Following expression (2.9), we write the stress tensor three-point function as

$$\langle T_{\mu\nu}(x)T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle = \Gamma_{\mu\nu\sigma\rho\alpha\beta}(x,y,z) - \frac{\mu^{\epsilon}}{2\pi^2\epsilon}\Big(cD^{W}_{\mu\nu\sigma\rho\alpha\beta}(x,y,z) - aD^{E}_{\mu\nu\sigma\rho\alpha\beta}(x,y,z)\Big), \tag{4.4}$$

where the μ-dependent part is determined by the counter-terms. We take

$$D^{E}_{\mu\nu\sigma\rho\alpha\beta}(x,y,z) = \lim_{g_{\mu\nu}\to\delta_{\mu\nu}} \frac{\delta^3}{\delta g^{\mu\nu}(x)\delta g^{\sigma\rho}(y)\delta g^{\alpha\beta}(z)}\int_{\mathcal{M}} d^dx'\sqrt{g}E_4(x'), \tag{4.5}$$

$$D^{W}_{\mu\nu\sigma\rho\alpha\beta}(x,y,z) = \lim_{g_{\mu\nu}\to\delta_{\mu\nu}} \frac{\delta^3}{\delta g^{\mu\nu}(x)\delta g^{\sigma\rho}(y)\delta g^{\alpha\beta}(z)}\int_{\mathcal{M}} d^dx'\sqrt{g}W^{2(d)}_{\mu\nu\lambda\rho}(x'). \tag{4.6}$$

In the dimensional regularization scheme, we consider a different treatment on the totally-antisymmetic tensor compared to [23, 24]. We write the Euler density as

$$\int_{\mathcal{M}} d^dx\sqrt{g}E_4 = \int_{\mathcal{M}} \frac{\left(\bigwedge_{j=1}^{d} dx^{\mu_j}\right)}{4(d-4)!} R^{a_1a_2}{}_{\mu_1\mu_2}R^{a_3a_4}{}_{\mu_3\mu_4}e^{a_5}_{\mu_5}\cdots e^{a_d}_{\mu_d}\epsilon_{a_1\cdots a_d}. \tag{4.7}$$

Defining $\sigma(x)$ as the Weyl transformation parameter, $g_{\mu\nu}(x)\to e^{2\sigma}g_{\mu\nu}(x)$, we have

$$\lim_{d\to 4}\frac{\delta}{(d-4)\delta\sigma(x)}\int_{\mathcal{M}} d^dx'\sqrt{g}\Big(E_4(x'), W^{2(d)}_{\mu\nu\lambda\rho}(x')\Big) = \sqrt{g}\Big(E_4(x), W^2_{\mu\nu\lambda\rho}(x)\Big), \tag{4.8}$$

($W^2_{\mu\nu\lambda\rho} \equiv W^{2(d=4)}_{\mu\nu\lambda\rho}$) which confirms that the counter-terms produce the Weyl anomaly.

Let us first consider a metric variation on (4.7). Performing integration by parts and recalling the metricity and the Bianchi identity, in varying the integrated Euler density we in fact only need to vary the vielbeins. From the relation $2\delta/\delta g^{\nu}_{\mu} = e^{a}_{(\nu}\delta/\delta e^{a}_{\mu)}$, we obtain

$$\frac{\delta}{\delta g^{\nu}_{\mu}(x)}\int_{\mathcal{M}} d^dx'\sqrt{g}E_4 = \frac{\sqrt{g}}{8}R^{\nu_1\nu_2}{}_{\mu_1\mu_2}R^{\nu_4\nu_4}{}_{\mu_3\mu_4}\,\delta^{\mu_1\cdots\mu_4\mu}_{\nu_1\cdots\nu_4\nu}. \tag{4.9}$$

After performing two more metric variations and imposing the flat space limit, the result is

$$D^{E}_{\mu\nu\sigma\rho\alpha\beta}(x,y,z) = -\frac{1}{4}\left(\epsilon_{\mu\sigma\alpha\eta\xi}\epsilon_{\nu\rho\beta\zeta\delta}\partial^{\eta}\partial^{\zeta}\delta^4(x-y)\partial^{\xi}\partial^{\delta}\delta^4(x-z) + \sigma\leftrightarrow\rho, \alpha\leftrightarrow\beta\right). \tag{4.10}$$

However, the a-charge does not contribute to the RG equation of the three-point function because $\lim_{d\to 4} D^{E}_{\mu\nu\sigma\rho\alpha\beta}(x,y,z) = 0$ due to the five totally anti-symmetized indices. It is the c-charge that controls the RG equation

$$\mu\frac{\partial}{\partial\mu}\langle T_{\mu\nu}(x)T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle = -\frac{c}{2\pi^2}\lim_{d\to 4} D^{W}_{\mu\nu\sigma\rho\alpha\beta}(x,y,z). \tag{4.11}$$

[2]It is shown recently that this type-D anomaly can be intrinsically fixed by supersymmetry [28].

While the explicit expression of $D^W_{\mu\nu\sigma\rho\alpha\beta}(x,y,z)$ is not essential for our purpose, we remark that, from the following re-writing

$$W^{2(d)}_{\mu\nu\lambda\rho} = E_4 + \frac{4(d-3)}{(d-2)} Q_d, \quad Q_d = R^2_{\mu\nu} - \frac{d}{4(d-1)} R^2\,, \tag{4.12}$$

one can also say the RG flow is dictated by the Q-curvature:[3]

$$D^W_{\mu\nu\sigma\rho\alpha\beta}(x,y,z) = 2 \lim_{g_{\mu\nu}\to\delta_{\mu\nu}} \left(\frac{\delta^3}{\delta g^{\mu\nu}(x)\delta g^{\sigma\rho}(y)\delta g^{\alpha\beta}(z)} \int_{\mathcal{M}} d^d x' \sqrt{g} Q_d(x') \right). \tag{4.13}$$

The above third-order metric variation can be performed straightforwardly. However, we do not find a compact expression so we do not list the result here.

Finally, given the conformal anomaly (4.1), the Ward identity (3.7) can also be computed and the result is given by [23]

$$\begin{aligned} &\langle T^\mu_\mu(x) T_{\sigma\rho}(y) T_{\alpha\beta}(z)\rangle \\ &= 2\Big(\delta^4(x-y) + \delta^4(z-x)\Big) \langle T_{\sigma\rho}(y) T_{\alpha\beta}(z)\rangle + \frac{1}{4\pi^2}\Big(c B_{\sigma\rho\alpha\beta}(x,y,z) - a A_{\sigma\rho\alpha\beta}(x,y,z)\Big). \end{aligned} \tag{4.14}$$

The anomalous terms are determined by the second-order expansion of the anomalies:

$$\begin{aligned} A_{\sigma\rho\alpha\beta}(x,y,z) &= \lim_{g_{\mu\nu}\to\delta_{\mu\nu}} \frac{\delta^2}{\delta g^{\sigma\rho}(y)\delta g^{\alpha\beta}(z)} E_4(x) \\ &= -\Big\{\epsilon_{\sigma\alpha\lambda\mu}\epsilon_{\rho\beta\delta\nu}\partial^\mu\partial^\nu\Big(\partial^\lambda\delta^4(x-y)\partial^\delta\delta^4(x-z)\Big) + \sigma \leftrightarrow \rho\Big\}\,, \end{aligned} \tag{4.15}$$

$$\begin{aligned} B_{\sigma\rho\alpha\beta}(x,y,z) &= \lim_{g_{\mu\nu}\to\delta_{\mu\nu}} \frac{\delta^2}{\delta g^{\sigma\rho}(y)\delta g^{\alpha\beta}(z)} W^2_{\mu\nu\lambda\delta}(x) \\ &= 8\Big(P_{\sigma\mu\nu\rho,\alpha\gamma\delta\beta}\partial^\mu\partial^\nu\delta^4(x-y)\partial^\gamma\partial^\delta\delta^4(x-z)\Big)\,, \end{aligned} \tag{4.16}$$

where a projector is introduced that shares the same symmetries as the Weyl tensor:

$$\begin{aligned} P_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta} = &\frac{1}{12}\big(\delta_{\mu\alpha}\delta_{\nu\beta}\delta_{\sigma\gamma}\delta_{\rho\delta} + \delta_{\mu\delta}\delta_{\sigma\beta}\delta_{\rho\alpha}\delta_{\nu\gamma} - \mu\leftrightarrow\sigma, \nu\leftrightarrow\rho\big) \\ &+ \frac{1}{24}\big(\delta_{\mu\alpha}\delta_{\nu\gamma}\delta_{\rho\delta}\delta_{\sigma\beta} - \mu\leftrightarrow\sigma, \nu\leftrightarrow\rho, \alpha\leftrightarrow\gamma, \delta\leftrightarrow\beta\big) \\ &- \frac{1}{16}\big(\delta_{\mu\rho}\delta_{\alpha\delta}\delta_{\sigma\gamma}\delta_{\nu\beta} + \delta_{\mu\rho}\delta_{\alpha\delta}\delta_{\sigma\beta}\delta_{\nu\gamma} - \mu\leftrightarrow\sigma, \nu\leftrightarrow\rho, \alpha\leftrightarrow\gamma, \delta\leftrightarrow\beta\big) \\ &+ \frac{1}{12}\Big(\delta_{\mu\rho}\delta_{\nu\sigma} - \rho\leftrightarrow\nu\big)\big(\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta\leftrightarrow\beta\Big). \end{aligned} \tag{4.17}$$

(To the first order in the metric expansion, one has $W_{\mu\sigma\rho\nu} \sim 2P_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}\partial^\gamma\partial^\delta\delta g^{\alpha\beta}$.) From the general statement [23] that there are three independent coefficients in the three-point function, the relation (4.14) implies that there exists a linear relation between the three coefficients and the central charges a and c. (Assume that the type-D anomaly is removed.)

[3] In $d = 4$ the notion of Q-curvature was introduced by Branson and Ørsted [29] as a global conformal invariant. However, note that the curvature Q_4 in (4.12) is different from the Q-curvature defined in [29] by a total derivative, $\Box R$. For an analysis of Q-curvature in $d = 6$, see [30].

4.2 Boundary terms of anomalies and effective action

The complete classification of possible boundary terms based on the Wess-Zumino consistency for $d = 4$ CFTs was carried out recently by [17]. Denoting the induced boundary metric as $h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$ where n_μ is the unit-length, outward-pointing normal vector to $\partial\mathcal{M}$, the full Weyl anomaly with boundary terms is given by

$$\langle T^\mu_\mu(x)\rangle = \frac{1}{16\pi^2}\Big(cW^2_{\mu\nu\lambda\rho} - aE_4\Big) + \frac{\delta(x_\perp)}{16\pi^2}\Big(aE_4^{(\mathrm{bry})} + b_1\,\mathrm{tr}\,\hat{K}^3 + b_2 h^{\alpha\gamma}\hat{K}^{\beta\delta}W_{\alpha\beta\gamma\delta}\Big)\,, \quad (4.18)$$

where $\delta(x_\perp)$ is a Dirac delta function with support on the boundary. The Chern-Simons-like boundary term of the Euler Characteristic,

$$\begin{aligned} E_4^{(\mathrm{bry})} &= -4\delta^{\mu_1\mu_2\mu_3}_{\nu_1\nu_2\nu_3}\; K^{\nu_1}_{\mu_1}\left(\frac{1}{2}R^{\nu_2\nu_3}{}_{\mu_2\mu_3} + \frac{2}{3}K^{\nu_2}_{\mu_2}K^{\nu_3}_{\mu_3}\right) \\ &= 4\left(2\mathring{E}_{\alpha\beta}K^{\alpha\beta} + \frac{2}{3}\mathrm{tr}K^3 - K\,\mathrm{tr}\,K^2 + \frac{1}{3}K^3\right)\,, \end{aligned} \quad (4.19)$$

is used to supplement the bulk density E_4 to preserve the topological invariance. (In the literature, it is also referred to as the boundary term for the Lovelock gravity. See [31] or [17] for a review.) We have denoted $\mathring{E}_{\alpha\beta}$ as the boundary Einstein tensor. There are two additional boundary Weyl invariants in the anomaly (4.18) and we refer to b_1 and b_2 as "boundary" central charges. The Weyl curvature in the last term of (4.18) and the Riemann curvature in (4.19) are pulled-back tensors. The traceless part of the extrinsic curvature is given by

$$\hat{K}_{\alpha\beta} = K_{\alpha\beta} - \frac{K}{d-1}h_{\alpha\beta}\,, \quad (4.20)$$

with $d = 4$ here. (We still adopt the Greek indices for these boundary tensors, but note that their normal component is empty.) It is an important property that $\hat{K}_{\alpha\beta}$ transforms covariantly under the Weyl scaling. The boundary b_1-type anomaly to our knowledge first appeared in [32]; [33] first pointed out the boundary b_2-type anomaly. Spelling out the expressions, we have

$$\mathrm{tr}\,\hat{K}^3 = \mathrm{tr}\,K^3 - K\,\mathrm{tr}\,K^2 + \frac{2}{9}K^3\,, \quad (4.21)$$

$$h^{\alpha\gamma}\hat{K}^{\beta\delta}W_{\alpha\beta\gamma\delta} = R^\mu{}_{\nu\lambda\rho}K^\lambda_\mu n^\nu n^\rho - \frac{1}{2}R_{\mu\nu}(n^\mu n^\nu K + K^{\mu\nu}) + \frac{1}{6}KR\,. \quad (4.22)$$

Notice that, because of these boundary terms, the Weyl anomaly (the log-divergent term of the partition function) of CFTs does not vanish even in flat space. In particular, the boundary b_1-type anomaly constructed solely from the extrinsic curvature exists in any dimensions, and it might play a role to order the (boundary) RG flows in any-dimensional QFTs.

The conformal anomaly has its origin tracing back to the local counter-terms. Adopting the dimensional regularization, we should include the corresponding boundary terms in the effective action. We have the following identities:

$$\lim_{d\to 4}\frac{\delta}{(d-4)\delta\sigma(x)}\left(\int_{\mathcal{M}} d^dx'\sqrt{g}\ E_4(x')-\int_{\partial\mathcal{M}} d^{d-1}x'\sqrt{h}\ E_4^{(\mathrm{bry})}(x')\right)=\sqrt{g}E_4-\sqrt{h}E_4^{(\mathrm{bry})}\,, \tag{4.23}$$

$$\lim_{d\to 4}\frac{\delta}{(d-4)\delta\sigma(x)}\int_{\partial\mathcal{M}} d^{d-1}x'\sqrt{h}\ \mathrm{tr}\,\hat{K}^3(x')=\sqrt{h}\,\mathrm{tr}\,\hat{K}^3\,, \tag{4.24}$$

$$\lim_{d\to 4}\frac{\delta}{(d-4)\delta\sigma(x)}\int_{\partial\mathcal{M}} d^{d-1}x'\sqrt{h}\ h^{\alpha\gamma}\hat{K}^{\beta\delta}W^{(d)}_{\alpha\beta\gamma\delta}(x')=\sqrt{h}h^{\alpha\gamma}\hat{K}^{\beta\delta}W^{(4)}_{\alpha\beta\gamma\delta}\,. \tag{4.25}$$

The first identity is the consequence of the topological invariance when the boundary term of the Euler Characteristic is included; the last two identities reflect that they are covariant Weyl tensors.

We find a compact expression for the boundary type-b_1 anomaly given by

$$\mathrm{tr}\,\hat{K}^3=\frac{1}{2}\delta^{\mu\nu\lambda}_{\rho\sigma\delta}\hat{K}^{\rho}_{\mu}\hat{K}^{\sigma}_{\nu}\hat{K}^{\delta}_{\lambda}\,. \tag{4.26}$$

using three Kronecker delta functions. In the dimensional regularization, we write

$$\int_{\partial\mathcal{M}} d^{d-1}x\sqrt{h}\,\mathrm{tr}\,\hat{K}^3=\int_{\partial\mathcal{M}}\frac{\left(\bigwedge_{j=1}^{d-1}dx^{\mu_j}\right)}{2(d-4)!}\hat{K}^{a_1}_{\mu_1}\hat{K}^{a_2}_{\mu_2}\hat{K}^{a_3}_{\mu_3}e^{a_4}_{\mu_4}\cdots e^{a_{d-1}}_{\mu_{d-1}}\epsilon_{a_1\cdots a_{d-1}}\,, \tag{4.27}$$

similar to the way we express the bulk Euler density in d dimensions using vielbeins in (4.7).

We will work in Gaussian normal coordinates, $x^\mu=\{x_\perp,x^i\}$, such that $x_\perp=0$ is the local function for the boundary. The metric is given by

$$ds^2=dx_\perp^2+h_{ij}(x_\perp,x^i)dx^idx^j\,. \tag{4.28}$$

Focusing on the response from varying the bulk metric of $\mathcal{M}$, we keep the boundary metric $h_{ij}(x_\perp=0,x^i)$ fixed while we perform the variation. However, the normal derivative of the metric variation on the boundary can be non-zero in general. Under the coordinate (4.28), on the boundary we require

$$\delta g_{\mu\nu}|_{\partial\mathcal{M}}=\delta h_{\mu\nu}(x_\perp=0,x^i)=0\,, \tag{4.29}$$

$$\partial_{\mathrm{n}}\delta g_{\mathrm{n}i}|_{\partial\mathcal{M}}=\partial_{\mathrm{n}}\delta g_{\mathrm{nn}}|_{\partial\mathcal{M}}=\delta n_\mu|_{\partial\mathcal{M}}=0\,, \tag{4.30}$$

$$\partial_{\mathrm{n}}\delta g_{ij}|_{\partial\mathcal{M}}=\partial_{\mathrm{n}}\delta h_{ij}(x_\perp,x^i)|_{\partial\mathcal{M}}\neq 0\,. \tag{4.31}$$

The metric variation of extrinsic curvatures on the boundary then can be written by[4]

$$\delta K_{\mu\nu}|_{\partial\mathcal{M}}=\frac{1}{2}h^{\lambda}_{\mu}h^{\rho}_{\nu}\partial_{\mathrm{n}}\delta g_{\lambda\rho}|_{\partial\mathcal{M}}\,,\ \delta K|_{\partial\mathcal{M}}=\frac{1}{2}h^{\lambda\sigma}\partial_{\mathrm{n}}\delta g_{\lambda\sigma}|_{\partial\mathcal{M}}\,. \tag{4.32}$$

We will not discuss the correlation functions of the boundary stress tensor obtained by varying the boundary metric, h. In this case the contribution instead comes from the

[4]Relevant metric variation of curvatures can be found in the appendix.

tangential fluctuation of the metric along the boundary, and it will be independent of the bulk stress tensor correlation functions considered here.

It is instructive to verify the identity (4.24) using expression (4.27). We should show

$$\sqrt{h}\,\mathrm{tr}\,\hat{K}^3 = \lim_{d\to 4}\frac{2}{d-4}g^\mu_\nu\frac{\delta}{\delta g^\mu_\nu}\int_{\partial\mathcal{M}}\frac{\left(\bigwedge_{j=1}^{d-1}dx^{\mu_j}\right)}{2(d-4)!}\hat{K}^{a_1}_{\mu_1}\hat{K}^{a_2}_{\mu_2}\hat{K}^{a_3}_{\mu_3}e^{a_4}_{\mu_4}\cdots e^{a_{d-1}}_{\mu_{d-1}}\epsilon_{a_1\cdots a_{d-1}}\,. \tag{4.33}$$

In the metric variation we receive contributions from varying the extrinsic curvatures and also from varying the vielbeins.[5] The latter procedure results in terms $\sim \hat{K}^\omega_\rho\hat{K}^\sigma_\eta\hat{K}^\lambda_\delta\delta^{\rho\eta\delta\mu}_{\omega\sigma\lambda\nu}$ and identity (4.24) can be recovered by contracting μ with ν. But one then needs to show that the contribution from varying the extrinsic curvatures with respect to metric is traceless. We can see this is indeed the case by writing the explicit metric expansion as

$$\hat{K}^\beta_\alpha \sim \left(K^\beta_\alpha + \frac{1}{2}h^\lambda_\alpha h^{\beta\rho}\partial_{\mathrm{n}}\delta g_{\lambda\rho}\right) - \frac{h^\beta_\alpha}{d-1}\left(K + \frac{1}{2}h^{\lambda\rho}\partial_{\mathrm{n}}\delta g_{\lambda\rho}\right) = \hat{K}^\beta_\alpha + \frac{1}{2}H_\alpha{}^{\lambda\beta\rho}\partial_{\mathrm{n}}\delta g_{\lambda\rho}\,, \tag{4.34}$$

where a tensor is introduced,

$$H_\alpha{}^{\lambda\beta\rho} = h^\lambda_\alpha h^{\beta\rho} - \frac{1}{d-1}h^\beta_\alpha h^{\lambda\rho}\,, \tag{4.35}$$

with the desired traceless property: $H_\alpha{}^{\lambda\alpha\rho} = H_{\alpha\lambda}{}^{\beta\lambda} = 0$. The higher-order expansions of (4.34) vanish due to the boundary condition. Therefore, all we need to include on the boundary is the first-order part.

4.3 Renormalization group flow with a boundary

Taking into account the boundary terms of the conformal anomaly (4.18), we would like to see how the RG equation (4.11) might be affected by boundary central charges. We shall focus on potential μ-dependent divergences coming from the boundary local counter-terms.

First we discuss the boundary term of the a-type anomaly. With the boundary term, the fact that the Lovelock gravity has a well-defined variational principle in the presence of a boundary implies that there will be no boundary contribution to the metric variation left over. In dimensions $d = 4+\epsilon$, the additional variation comes from varying the vielbeins, which only appear through the wedge products. Varying these vielbeins gives additional indices in the totally antisymmetric tensor that vanishes in the limit $d \to 4$.[6] In short, the topological nature of Euler characteristic is still preserved in a non-compact manifold and the RG flow remains to be independent of the a-charge. The argument applies to any even dimensions.

The story is more subtle for the boundary Weyl invariants. Let us first consider the boundary b_2-type. In a recent paper [34], it is shown that the metric variation of the

[5] We impose the boundary condition on the metric only in the physical dimensions. In the dimensional regularization scheme, the extra dimensions introduce additional vielbeins, whose variation on the boundary should be kept non-vanishing. It is the variation of these vielbeins that allows us to correctly reproduce the anomaly by varying an effective action.

[6] For certain geometries one could obtain finite stress tensors. We will discuss them in the next section. However, for the RG equation only the μ-dependent poles are relevant.

following action:

$$S = \int_{\mathcal{M}} d^d x\sqrt{g}\; W^n_{\alpha\beta\mu\nu} - n\int_{\partial\mathcal{M}} d^{d-1}x\sqrt{h}\; P^{\alpha\beta\mu\nu}W^{n-1}_{\alpha\beta\mu\nu}\,, \tag{4.36}$$

does not have boundary terms containing normal derivatives of the metric variation left over. The tensor $P_{\alpha\beta\mu\nu}$ has the same symmetries as the $W_{\alpha\beta\mu\nu}$ and is defined by

$$P_{\alpha\beta\mu\nu} = P^{(0)}_{\alpha\beta\mu\nu} - \frac{\left(g_{\alpha\mu}P^{(0)}_{\beta\nu} - g_{\alpha\nu}P^{(0)}_{\beta\mu} - g_{\beta\mu}P^{(0)}_{\alpha\nu} + g_{\beta\nu}P^{(0)}_{\alpha\mu}\right)}{d-2} + \frac{P^{(0)}(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu})}{(d-1)(d-2)}\,, \tag{4.37}$$

where

$$P^{(0)}_{\alpha\beta\mu\nu} = n_\alpha n_\nu K_{\beta\mu} - n_\beta n_\nu K_{\alpha\mu} - n_\alpha n_\mu K_{\beta\nu} + n_\beta n_\mu K_{\alpha\nu}\,, \tag{4.38}$$

$$P^{(0)}_{\beta\nu} = g^{\alpha\mu}P^{(0)}_{\alpha\beta\mu\nu} = -n_\beta n_\nu K - K_{\beta\nu}\,, \tag{4.39}$$

$$P^{(0)} = g^{\beta\nu}P^{(0)}_{\beta\nu} = -2K\,. \tag{4.40}$$

Note that $P^{\alpha}_{\ \mu\alpha\nu} = 0$. Under the conformal transformation, $P_{\alpha\beta\mu\nu} \to e^{3\sigma}P_{\alpha\beta\mu\nu}$. To relate this formulation with the effective action of the type-B anomaly, we take $n = 2$ in (4.36). Using $\mathrm{tr}(PW) = \mathrm{tr}(P^{(0)}W) = 4\hat{K}_{\nu\alpha}W^{\mu\nu\alpha\beta}n_\mu n_\beta$ we write

$$\tilde{W} = \frac{c}{16\pi^2}\left(\int_{\mathcal{M}} d^d x\sqrt{g}\; W^{2(d)}_{\mu\nu\lambda\rho} + 8\int_{\partial\mathcal{M}} d^{d-1}x\sqrt{h}\; h_{\mu\beta}\hat{K}_{\nu\alpha}W^{(d)\mu\nu\beta\alpha}\right). \tag{4.41}$$

The effective action then might be defined by multiplying the above result by the $\frac{\mu^\epsilon}{\epsilon}$ factor. It is interesting to notice that the coefficient "8" of the boundary term suggests $b_2 = 8c$ [34].

There are, however, some suspicions about this derivation that leads to the relation $b_2 = 8c$, which would mean that b_2 is not an independent central charge. First of all, from the classical viewpoint, in order to integrate the equations of motion of Weyl gravity, one needs to specify more boundary data. The theory of Weyl gravity might be still consistent (at the classical level) even if on the boundary there are some normal derivatives of the metric variation left over. (The Lovelock gravity is special in that the action is quadratic in time derivatives.) On the other hand, at the quantum level, it becomes not clear that the anomaly effective action must have a nice variational principle. In other words, one might allow some (traceless) delta function distributions on the boundary as a part of the contribution to the type-B anomaly induced stress tensor.

Because of these potential issues, one might adopt the following decomposition of the type-b anomalies related effective action:

$$\widetilde{W}_B = \frac{\mu^\epsilon}{\epsilon}\left[\frac{c}{16\pi^2}\left(\int_{\mathcal{M}} W^{2(d)}_{\mu\nu\lambda\rho} + 8\int_{\partial\mathcal{M}} h_{\mu\beta}\hat{K}_{\nu\alpha}W^{(d)\mu\nu\beta\alpha}\right)\right.$$
$$\left. + \frac{1}{16\pi^2}\int_{\partial\mathcal{M}}\left(b_1\,\mathrm{tr}\,\hat{K}^3 + b_2' h_{\mu\beta}\hat{K}_{\nu\alpha}W^{(d)\mu\nu\beta\alpha}\right)\right]. \tag{4.42}$$

From the conformal invariance requirement, we are allowed to add more $h\hat{K}W$ on the boundary with the coefficient measuring the difference $b_2' = b_2 - 8c$. Indeed, if the argument using the variational principle is invalid, one needs another derivation to verify b_2'. However, we notice that the spin $0, \frac{1}{2}, 1$ case's free field calculations given recently in [35] suggest the following universal result (independent of boundary conditions)

$$b_2' = 0\,. \tag{4.43}$$

This is the scenario we adopt in what follows. It is still interesting, though, to find a general proof of the result (4.43) without referring to the variational method.

Let us go back to the correlation function. As the consequence of the decomposition (4.42) and the result (4.43), the b_2-type boundary term basically plays the role of a Gibbons-Hawking-like term for the bulk type-B anomaly effective action. Notice that (4.41) applies directly in $d = 4{+}\epsilon$ dimensions. Therefore, this boundary b_2-type does not generate a μ-dependent pole in the three-point function, and hence the RG flow is not touched by the b_2-charge.

Finally we consider the boundary b_1-type. We will see this type of boundary Weyl invariant contributes to the RG equation in the vicinity of the boundary. This central charge depends on boundary conditions; see [35] for related heat kernel computation. Focusing on the μ-dependent, singular contribution in the $d \to 4$ limit, we denote the correction as

$$\Delta\langle T_{\mu\nu}(x)T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle^{(b_1)} = -\frac{b_1}{2\pi^2}\frac{\mu^\epsilon}{\epsilon} D^{(b_1)}_{\mu\nu\sigma\rho\alpha\beta}(x,y,z)\,, \tag{4.44}$$

where

$$D^{(b_1)}_{\mu\nu\sigma\rho\alpha\beta}(x,y,z) = \frac{\delta^3}{\delta g^{\mu\nu}(x)\delta g^{\sigma\rho}(y)\delta g^{\alpha\beta}(z)} \int_{\partial\mathcal{M}} d^{d-1}x\sqrt{h}\,\mathrm{tr}\,\hat{K}^3\,. \tag{4.45}$$

Using expression (4.27), we obtain

$$\lim_{d\to 4} D^{(b_1)}_{\mu\nu\sigma\rho\alpha\beta}(x,y,z) = -\frac{3}{8}\delta^{\delta\omega\lambda}_{\eta\zeta\xi} H_{\delta(\mu}{}^{\eta}{}_{\nu)}(x) H_{\omega(\sigma}{}^{\zeta}{}_{\rho)}(x) H_{\lambda(\alpha}{}^{\xi}{}_{\beta)}(x)\partial_{\rm n}\delta(x_\perp)\partial_{\rm n}\delta^4(x{-}y)\partial_{\rm n}\delta^4(x{-}z)\,. \tag{4.46}$$

The normal component does not contribute and the correction only exists near the boundary.

In summary, as the generalization of (4.11), the RG equation of the three-point function for $d = 4$ QFTs in a flat manifold with a boundary is given by

$$\mu\frac{\partial}{\partial\mu}\langle T_{\mu\nu}(x)T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle = -\frac{c}{2\pi^2}\lim_{d\to 4} D^{W}_{\mu\nu\sigma\rho\alpha\beta}(x,y,z) - \frac{b_1}{2\pi^2}\lim_{d\to 4} D^{(b_1)}_{\mu\nu\sigma\rho\alpha\beta}(x,y,z)\,. \tag{4.47}$$

$D^{W}_{\mu\nu\sigma\rho\alpha\beta}$ is determined by the Q-curvature via (4.13) and the boundary correction is given in (4.46).

4.4 Conformal ward identity with a boundary and stress tensor

We next revisit the conformal Ward identity by including boundary anomalies. As the generalization of (4.14), we find

$$
\begin{aligned}
&\langle T^{\mu}_{\mu}(x)T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle \\
&=2\Big(\delta^4(x-y)+\delta^4(z-x)\Big)\langle T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle+\frac{1}{4\pi^2}\Big(cB_{\sigma\rho\alpha\beta}(x,y,z)-aA_{\sigma\rho\alpha\beta}(x,y,z)\Big) \\
&\quad+\frac{\delta(x_\perp)}{4\pi^2}\Big(aE_{\sigma\rho\alpha\beta}(x,y,z)+b_1 b^{(1)}_{\sigma\rho\alpha\beta}(x,y,z)+b_2 b^{(2)}_{\sigma\rho\alpha\beta}(x,y,z)\Big) \\
&\quad-\frac{a}{2\pi^2}\delta(x_\perp)\delta^{\mu\nu\lambda}_{\delta\eta\zeta}K^{\delta}_{\mu}(x)K^{\eta}_{\nu}(x)\Big(\delta_{\alpha\beta}h^{\zeta}_{(\rho}h_{\lambda|\sigma)}(x)\delta^4(x-z)\partial_{\rm n}\delta^4(x-y)+\sigma\leftrightarrow\alpha,\rho\leftrightarrow\beta,y\leftrightarrow z\Big) \\
&\quad+\frac{3b_1}{16\pi^2}\delta(x_\perp)\delta^{\mu\nu\lambda}_{\rho\sigma\delta}\hat{K}^{\rho}_{\mu}(x)\hat{K}^{\sigma}_{\nu}(x)\Big(\delta_{\alpha\beta}H_{\lambda(\sigma}{}^{\delta}{}_{\rho)}(x)\delta^4(x-z)\partial_{\rm n}\delta^4(x-y)+\sigma\leftrightarrow\alpha,\rho\leftrightarrow\beta,y\leftrightarrow z\Big) \\
&\quad+S_{\sigma\rho\alpha\beta}(x,y,z)\langle T^{\mu}_{\mu}(x)\rangle|_{\delta_{\mu\nu}}\,,
\end{aligned}
\tag{4.48}
$$

where

$$
\langle T^{\mu}_{\mu}(x)\rangle|_{\delta_{\mu\nu}}=\frac{\delta(x_\perp)}{16\pi^2}\Big(b_1\,\mathrm{tr}\,\hat{K}^3-\frac{8}{3}a\delta^{\mu\nu\lambda}_{\sigma\rho\delta}K^{\sigma}_{\mu}K^{\rho}_{\nu}K^{\delta}_{\lambda}\Big)\,,
\tag{4.49}
$$

is the flat limit of the conformal anomaly. The results $A_{\sigma\lambda\rho\eta}(x,y,z)$ and $B_{\sigma\lambda\rho\eta}(x,y,z)$ are given by (4.15) and (4.16), respectively; the tensor $S_{\sigma\rho\alpha\beta}$ is defined in (3.8). We also have

$$
\begin{aligned}
E_{\sigma\rho\alpha\beta}(x,y,z)&=\lim_{g_{\mu\nu}\to\delta_{\mu\nu}}\frac{\delta^2}{\delta g^{\alpha\beta}(z)\delta g^{\sigma\rho}(y)}E^{(\rm bry)}_4(x) &(4.50)\\
&=-4\;\delta^{\mu_1\mu_2\mu_3}_{\nu_1\nu_2\nu_3}\;K^{\nu_1}_{\mu_1}(x)h^{\nu_2}_{(\beta}h_{\mu_2|\alpha)}(x)h^{\nu_3}_{(\rho}h_{\mu_3|\sigma)}(x)\partial_{\rm n}\delta^4(x-y)\partial_{\rm n}\delta^4(x-z)\,,\\
b^{(1)}_{\sigma\rho\alpha\beta}(x,y,z)&=\lim_{g_{\mu\nu}\to\delta_{\mu\nu}}\frac{\delta^2}{\delta g^{\alpha\beta}(z)\delta g^{\sigma\rho}(y)}\,\mathrm{tr}\,\hat{K}^3(x) &(4.51)\\
&=\frac{3}{4}\;\delta^{\mu_1\mu_2\mu_3}_{\nu_1\nu_2\nu_3}\hat{K}^{\nu_1}_{\mu_1}(x)H_{\mu_1(\alpha}{}^{\nu_2}{}_{\beta)}(x)H_{\mu_3(\sigma}{}^{\nu_3}{}_{\rho)}(x)\partial_{\rm n}\delta^4(x-y)\partial_{\rm n}\delta^4(x-z)\,,\\
b^{(2)}_{\sigma\rho\alpha\beta}(x,y,z)&=\lim_{g_{\mu\nu}\to\delta_{\mu\nu}}\frac{\delta^2}{\delta g^{\alpha\beta}(z)\delta g^{\sigma\rho}(y)}h^{\alpha\gamma}\hat{K}^{\beta\delta}W_{\alpha\beta\gamma\delta}(x)=0\,. &(4.52)
\end{aligned}
$$

We have used the fact that when the Weyl tensor or the Riemann tensor is pulled-back on the boundary, there are no normal derivatives acting on the metric variation left over.

Notice that there is a μ-dependent pole from the two-point function proportional to the c-charge. As noticed by [23], this singular behaviour can be reproduced (or derived from definition (3.2)) by requiring compatibility between the three point function and the Ward identity (3.6). The result is given by

$$
\langle T_{\mu\nu}(x)T_{\sigma\rho}(y)\rangle^{(c)}\sim-\frac{c}{4\pi^2}\frac{\mu^{\epsilon}}{\epsilon}\Delta^{T}_{\mu\nu\sigma\rho}\delta^4(x-y),\quad \Delta^{T}_{\mu\nu\sigma\rho}=\frac{1}{2}(S_{\mu\sigma}S_{\nu\rho}+\sigma\leftrightarrow\rho)-\frac{1}{3}S_{\mu\nu}S_{\sigma\rho}\,,
\tag{4.53}
$$

where $S_{\mu\nu}=(\partial_\mu\partial_\nu-\delta_{\mu\nu}\Box)$. (For the contact term, see (2.6).) Similarly, the boundary charge b_1 also contributes a pole to the two-point function. We have

$$
\langle T_{\mu\nu}(x)T_{\sigma\rho}(y)\rangle^{(b_1)}\sim\frac{3b_1}{16\pi^2}\frac{\mu^{\epsilon}}{\epsilon}\delta^{\lambda\eta\zeta}_{\alpha\beta\gamma}\hat{K}^{\alpha}_{\lambda}(x)H_{\eta(\mu}{}^{\beta}{}_{\nu)}(x)H_{\zeta(\sigma}{}^{\gamma}{}_{\rho)}(x)\partial_{\rm n}\delta(x_\perp)\partial_{\rm n}\delta^4(x-y)\,.
\tag{4.54}
$$

The correction does not have the normal component and only exists near the boundary.

Let us also discuss the expectation value of the stress tensor in the flat limit. Normally the stress tensor vanishes in flat space, but here a curved boundary generating boundary Weyl anomalies can lead to non-vanishing stress tensors.

The boundary b_2-type anomaly, as we have discussed earlier, does not touch the resulting metric variation because of the Gibbons-Hawking mechanism (we take $b_2 = 8c$). Therefore, there is no stress tensor correction by the b_2-charge.

Next we consider the contribution from the boundary b_1-type. We find

$$\langle T_\alpha^\beta(x)\rangle^{(b_1)} = -\frac{3b_1}{32\pi^2}\frac{\mu^\epsilon}{\epsilon}\delta^{\sigma\rho\eta}_{\mu\nu\lambda}\hat{K}^\mu_\sigma(x)\hat{K}^\nu_\rho(x)H_{\eta(\alpha}{}^{\lambda\beta)}(x)\partial_{\rm n}\delta(x_\perp) + \frac{b_1}{16\pi^2}\delta(x_\perp)t_\alpha^\beta(x)\,, \quad (4.55)$$

where the second term comes from varying the vielbeins and is given by

$$t_\alpha^\beta = \frac{\mu^\epsilon}{\epsilon}\frac{1}{2}\hat{K}^\rho_\mu\hat{K}^\sigma_\nu\hat{K}^\delta_\lambda\delta^{\mu\nu\lambda\beta}_{\rho\sigma\delta\alpha}\,. \quad (4.56)$$

Notice that the metric h^μ_ν showing up in $\hat{K}^\mu_\nu$ contracts with the generalized delta function and it then generates the factor $\epsilon = d-4$, cancelling the pole. We have

$$\begin{aligned} t_\alpha^\beta =& \frac{\mu^\epsilon}{\epsilon}\frac{1}{2}K^\rho_\mu K^\sigma_\nu K^\delta_\lambda\delta^{\mu\nu\lambda\beta}_{\rho\sigma\delta\alpha} \\ &-\frac{1}{2}\left(\frac{3K}{(d-1)}K^\rho_\mu K^\sigma_\nu\delta^{\mu\nu\beta}_{\rho\sigma\alpha} - 3\left(\frac{K}{(d-1)}\right)^2(d-3)\,K^\rho_\mu\delta^{\mu\beta}_{\rho\alpha} + \left(\frac{K}{(d-1)}\right)^3(d-3)\,(d-2)\delta^\beta_\alpha\right)|_{d\to4} \\ =& \frac{\mu^\epsilon}{\epsilon}\frac{1}{2}K^\rho_\mu K^\sigma_\nu K^\delta_\lambda\delta^{\mu\nu\lambda\beta}_{\rho\sigma\delta\alpha} - \frac{1}{2}\left(KK^\rho_\mu K^\sigma_\nu\delta^{\mu\nu\beta}_{\rho\sigma\alpha} - \frac{1}{3}K^2K^\rho_\mu\delta^{\mu\beta}_{\rho\alpha} + \frac{2}{27}K^3\delta^\beta_\alpha\right). \end{aligned} \quad (4.57)$$

One might argue that the first term simply vanishes in $(d-1)\to 3$ (the physical dimensionality of the boundary) limit, due to the additional indices in the totally-antisymmetric tensor. However, if the boundary geometry has a special form of the extrinsic curvature, for example, $K^\alpha_\beta = \frac{c}{r}h^\alpha_\beta$ where c is a constant, the contraction of indices can still lead to a finite result. In other words, the expression of the finite part of the stress tensor depends on the structure of the boundary.[7] For a ball where $\hat{K}^\alpha_\beta$ vanishes in d-dimensions, the stress tensor (4.55) vanishes directly. This type of boundary anomaly does not exist for the ball-like geometry.

The cylinder however provides a non-trivial example. The corresponding metric and extrinsic curvatures are given by

$$ds^2 = dr^2 + h_{ij}(r,x^i)dx^idx^j = dr^2 + \left(d^2z + r^2d^2\theta + r^2\sin^2\theta d^2\phi + \dots\right), \quad (4.58)$$

$$K^\nu_\mu = \mathrm{diag}\left(0,0,\frac{1}{r},\frac{1}{r},\dots\right), \qquad K = \frac{d-2}{r}\,. \quad (4.59)$$

It is useful to denote a delta function $\bar{\delta}^\mu_\nu = \mathrm{diag}\left(0,0,1,1,1,1,\dots\right)$ where r and z components are empty. The extrinsic curvature then can be written as $K^\nu_\mu = \frac{1}{r}\bar{\delta}^\nu_\mu$. We consider

[7] Similarly, the conformal flatness condition on the bulk geometry is implemented in [26] in order to have a finite bulk stress tensor in curved space.

a tube that has a radius r_0 so the boundary is set by $\delta(x_\perp) = \delta(r - r_0)$. The first term of (4.57) gives

$$\begin{aligned} K^\rho_\mu K^\sigma_\nu K^\delta_\lambda \delta^{\mu\nu\lambda\beta}_{\rho\sigma\delta\alpha} &= h^\beta_\alpha\Big(K^3 - 3K\,\mathrm{tr}\,K^2 + 2\,\mathrm{tr}\,K^3\Big) - 3K^\beta_\alpha\Big(K^2 - \mathrm{tr}\,K^2\Big) + 6K^\rho_\alpha\Big(KK^\beta_\rho - K^\sigma_\rho K^\beta_\sigma\Big) \\ &= \frac{1}{r_0^3}\Big(h^\beta_\alpha(d-4)(d-3)(d-2) - 3\bar{\delta}^\beta_\alpha(d-4)(d-3)\Big)\,, \end{aligned} \tag{4.60}$$

which generates an overall $d-4$ factor cancelling the pole. The second term of (4.57) evaluated in $d = 4$ contains the following contribution:

$$\Big(KK^\rho_\mu K^\sigma_\nu \delta^{\mu\nu\beta}_{\rho\sigma\alpha} - \frac{1}{3}K^2 K^\rho_\mu \delta^{\mu\beta}_{\rho\alpha} + \frac{2}{27}K^3\delta^\beta_\alpha\Big)|_{d\to 4} = \Big(\frac{52}{27}h^\beta_\alpha - \frac{8}{3}\bar{\delta}^\beta_\alpha\Big)\frac{1}{r_0^3}\,. \tag{4.61}$$

From (4.57), (4.60) and (4.61), by taking $d \to 4$ we obtain $t^\beta_\alpha = \frac{1}{r_0^3}\mathrm{diag}(0, \frac{1}{27}, -\frac{7}{54}, -\frac{7}{54})$. Next we consider the first piece in (4.55). Recalling the traceless property, we can write

$$\delta^{\sigma\rho\eta}_{\mu\nu\lambda}\hat{K}^\mu_\sigma \hat{K}^\nu_\rho H_{\eta(\alpha}{}^{\lambda\beta)} = 2\hat{K}^\nu_\mu \hat{K}^\mu_\rho H_{\nu(\alpha}{}^{\rho\beta)} = -\frac{2}{3r_0^2}\bar{\delta}^\mu_\nu H_{\mu(\alpha}{}^{\nu\beta)}\,, \tag{4.62}$$

where in the last equality we have used the cylindrical geometry. However, we see that this non-vanishing result does not generate a $d-4$ factor, and therefore it leads to an infinite contribution to the stress tensor. To have better behaviour, we add the following regulator:

$$\widetilde{W}^{\mathrm{reg}} = \frac{\mu^\epsilon}{\epsilon}c' \int_{\partial\mathcal{M}} d^3x\sqrt{h}\ \mathrm{tr}\,\hat{K}^3\,, \tag{4.63}$$

with the coefficient $c' = -\frac{b_1}{16\pi^2}$ being adjusted to cancel the divergence when taking $\epsilon \to 0$.[8] This regulator does not touch the anomaly coefficient since it is manifestly Weyl invariant. We obtain the final (renormalized) stress tensor

$$\langle T^\beta_\alpha\rangle^{(b_1)}|_{\mathrm{cylinder}} = \frac{b_1}{16\pi^2 r_0^3}\mathrm{diag}\Big(0, \frac{1}{27}, -\frac{7}{54}, -\frac{7}{54}\Big)\delta(r - r_0)\,. \tag{4.64}$$

The stress tensor contributes near the boundary and there is no r-component contribution, as expected. Taking the trace on this stress tensor gives

$$\langle T^\alpha_\alpha\rangle^{(b_1)}|_{\mathrm{cylinder}} = -\frac{b_1}{72\pi^2 r_0^3}\delta(r - r_0)\,, \tag{4.65}$$

which reproduces the Weyl anomaly evaluated for a $d = 4$ cylinder where

$$E_4^{(\mathrm{bry})} = 0, \quad \mathrm{tr}\,\hat{K}^3 = -\frac{2}{9r_0^3}\,. \tag{4.66}$$

We have mentioned that the Euler characteristic boundary term provides the Gibbons-Hawking mechanism so that the RG flow is untouched by the boundary counter-term $a\int_{\partial\mathcal{M}} E^{(\mathrm{bry})}$. However, there can be finite contribution to the stress tensor from varying

[8]Similar regulators are needed in order to have well-defined type-B anomaly induced stress tensors in a generally non-conformally flat background [27].

the vielbeins. (Varying the vielbeins on the bulk Euler characteristic counter-term gives the a-type stress tensor in curved space [26] and the result vanishes in the flat limit.) Including the boundary term, the Euler characteristic gives the following contribution to the stress tensor in the flat limit

$$\langle T_\alpha^\beta(x)\rangle^{(a)} = -\frac{a}{6\pi^2}\frac{\mu^\epsilon}{\epsilon}\delta(x_\perp)\delta^{\sigma\rho\eta\beta}_{\mu\nu\lambda\alpha}K^\mu_\sigma(x)K^\nu_\rho(x)K^\lambda_\eta(x)\,. \tag{4.67}$$

Let us first consider a ball with a radius r_0. Starting with bulk dimensionality d, we have

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 + r^2\sin^2\theta\sin^2\phi d\psi^2 + \dots\,, \tag{4.68}$$

$$K^\alpha_\beta = \frac{h^\alpha_\beta}{r}, \qquad K = \frac{(d-1)}{r}\,. \tag{4.69}$$

We obtain a finite result

$$\begin{aligned}\langle T^\beta_\alpha\rangle^{(a)}_{(\text{ball})} &= -\frac{a}{6\pi^2 r_0^3}\frac{\mu^\epsilon}{\epsilon}(d-4)(d-3)(d-2)\delta(r-r_0)\delta^\beta_\alpha \\ &= -\frac{a}{3\pi^2 r_0^3}\text{diag}\Big(0,1,1,1\Big)\delta(r-r_0)\,,\end{aligned} \tag{4.70}$$

where we have sent $d \to 4$ in the last equality. The trace of this stress tensor is

$$\langle T^\alpha_\alpha\rangle^{(a)}_{(\text{ball})} = -\frac{a}{\pi^2 r_0^3}\delta(r-r_0)\,, \tag{4.71}$$

which recovers the Weyl anomaly for a 4-ball where

$$E_4^{(\text{bry})} = -\frac{16}{r_0^3}, \quad \hat{K}_{\alpha\beta} = 0\,. \tag{4.72}$$

Let us next consider a cylinder. Notice that $E_4^{(\text{bry})}$ vanishes for a 4-cylinder and the anomaly comes solely from the boundary b_1-charge. One might think that, because of the vanishing $E_4^{(\text{bry})}$, there should not have any a-type stress tensor correction near the boundary. However, it is important to recall that the geometry can be fixed only after working out the variation. We find

$$\langle T^\beta_\alpha\rangle^{(a)}|_{\text{cylinder}} = \frac{a}{6\pi^2 r_0^3}\text{diag}\Big(0,-2,1,1\Big)\delta(r-r_0)\,, \tag{4.73}$$

for a 4-cylinder. The result is traceless, as expected.

It is of great interest to reproduce these new stress tensors in the vicinity of a boundary whose values are determined by boundary central charges using a different approach.

5 Three-dimensional CFTs

5.1 Boundary Weyl anomalies

Folklore has it that there is no Weyl anomaly in an odd dimension. This statement is based on the fact that it is impossible to construct a scalar with the correct dimension

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using curvatures. However, for an odd-dimensional manifold with a boundary, there can be Weyl anomalies localizing on the boundary.

For a $d = 3$ manifold, the anomaly is given by

$$\langle T^\mu_\mu(x)\rangle = \frac{a_3}{384\pi}\delta(x_\perp)\mathring{R} + \frac{c_3}{256\pi}\delta(x_\perp)\,\mathrm{tr}\,\hat{K}^2\,. \tag{5.1}$$

We denote $\mathring{R}$ as the intrinsic Ricci scalar on the boundary. The trace-free extrinsic curvature in this case is given by

$$\hat{K}_{\mu\nu} = K_{\mu\nu} - \frac{K}{2}h_{\mu\nu}\,. \tag{5.2}$$

In this notation, we have $(a_3, c_3) = (-1, 1)$ for a conformal scalar with the Dirichlet boundary condition and $(a_3, c_3) = (1, 1)$ for the conformal Robin condition [36]. (See [34] for a recent discussion on $d = 5$ CFTs.) In this section we discuss how these boundary charges contribute to stress tensor correlation functions.

5.2 Correlation function and stress tensor

We again focus on the correlation functions obtained by the bulk metric variation. We will not perturb the boundary metric, h, and the boundary conditions are (4.29), (4.30) and (4.31). The following identity relates the boundary counter-term to the a_3-type anomaly:

$$\sqrt{h}\mathring{R} = \lim_{d\to3}\frac{2}{d-3}g^\mu_\nu\frac{\delta}{\delta g^\mu_\nu}\int_{\partial\mathcal{M}}\frac{\left(\bigwedge_{j=1}^{d-1}dx^{\mu_j}\right)}{2(d-3)!}\mathring{R}^{a_1a_2}{}_{\mu_1\mu_2}e^{a_3}_{\mu_3}\cdots e^{a_{d-1}}_{\mu_{d-1}}\epsilon_{a_1\cdots a_{d-1}}\,. \tag{5.3}$$

The bulk metric variation does not touch the boundary Riemann curvature $\mathring{R}^{a_1a_2}{}_{\mu_1\mu_2}$ which is intrinsic on $\partial\mathcal{M}$. The contribution comes only from varying the vielbeins. The a_3-charge does not generate a μ-dependent pole, and hence the RG flows of (bulk) stress tensor correlation functions are independent from this boundary central charge.

For the boundary Weyl invariant, $\mathrm{tr}\,\hat{K}^2$, we first re-write it as

$$\mathrm{tr}\,\hat{K}^2 = -\delta^{\mu\nu}_{\alpha\beta}\hat{K}^\alpha_\mu\hat{K}^\beta_\nu\,. \tag{5.4}$$

Working now in $d = 3 + \epsilon$ dimensions, we have the following identity:

$$\sqrt{h}\,\mathrm{tr}\,\hat{K}^2 = \lim_{d\to3}\frac{-2}{d-3}g^\mu_\nu\frac{\delta}{\delta g^\mu_\nu}\int_{\partial\mathcal{M}}\frac{\left(\bigwedge_{j=1}^{d-1}dx^{\mu_j}\right)}{(d-3)!}\hat{K}^{a_1}_{\mu_1}\hat{K}^{a_2}_{\mu_2}e^{a_3}_{\mu_3}\cdots e^{a_{d-1}}_{\mu_{d-1}}\epsilon_{a_1\cdots a_{d-1}}\,, \tag{5.5}$$

that relates this conformal anomaly to the boundary counter-term. Using the general expression (3.7), we obtain the Ward identity

$$\begin{aligned}
&\langle T^\mu_\mu(x)T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle \\
&= 2(\delta^3(x-y)+\delta^3(x-z))\langle T_{\sigma\rho}(y)T_{\alpha\beta}(z)\rangle \\
&\quad - \frac{c_3}{128\pi}\delta(x_\perp)\delta^{\mu\nu}_{\lambda\delta}H_{\mu(\sigma}{}^{\lambda}{}_{\rho)}(x)H_{\nu(\alpha}{}^{\delta}{}_{\beta)}(x)\partial_{\mathrm{n}}\delta^3(x-y)\partial_{\mathrm{n}}\delta^3(x-z) \\
&\quad - \frac{c_3}{128\pi}\delta(x_\perp)\delta^{\mu\nu}_{\lambda\delta}\hat{K}^\lambda_\mu(x)\Big(H_{\nu(\sigma}{}^{\delta}{}_{\rho)}(x)\delta_{\alpha\beta}\delta^3(x-z)\partial_{\mathrm{n}}\delta^3(x-y) + \alpha\leftrightarrow\sigma, \beta\leftrightarrow\rho, y\leftrightarrow z\Big) \\
&\quad + S_{\alpha\beta\sigma\rho}\langle T^\mu_\mu(x)\rangle|_{\delta_{\mu\nu}}\,,
\end{aligned} \tag{5.6}$$

where the flat limit of the anomaly is the same as (5.1). We have defined $S_{\alpha\beta\sigma\rho}$ in (3.8).

We can also compute the RG equations of the correlation functions. The third-order metric expansion on the c_3-type effective action (which contains only two $\hat{K}_{\mu\nu}$ here) does not generate a μ-dependent pole, and therefore there is no correction to the RG equation of the three-point function. For the two-point function, a relevant contribution comes from varying the extrinsic curvatures. (Varying the vielbeins will not give a μ-dependent pole.) The RG flow is given by

$$\mu\frac{\partial}{\partial\mu}\langle T_{\mu\nu}(x)T_{\alpha\beta}(y)\rangle = \frac{c_3}{128\pi}\Big(\delta^{\lambda\rho}_{\sigma\delta}H_{\lambda(\alpha}{}^{\sigma}{}_{\beta)}(x)H_{\rho(\mu}{}^{\delta}{}_{\nu)}(x)\partial_{\rm n}\delta(x_\perp)\partial_{\rm n}\delta^3(x-y)\Big)\,. \tag{5.7}$$

This result applies for $d=3$ QFTs in a flat manifold with a boundary. The correction appears only near the boundary.

Let us next discuss stress tensors in $d=3$ flat space with a boundary. From the c_3-type anomaly, we have

$$\langle T^\nu_\mu(x)\rangle^{(c_3)} = \frac{c_3}{128\pi}\frac{\mu^\epsilon}{\epsilon}\delta^{\lambda\rho}_{\sigma\delta}\hat{K}^\sigma_\lambda(x)H_{\rho(\mu}{}^{\delta\nu)}(x)\partial_{\rm n}\delta(x_\perp) - \frac{c_3}{128\pi}\delta(x_\perp)t^\nu_\mu(x)\,. \tag{5.8}$$

Notice that $\delta^{\lambda\rho}_{\sigma\delta}h^\sigma_\lambda H_{\rho(\mu}{}^{\delta}{}_{\nu)} \sim H_{\rho(\mu}{}^{\rho}{}_{\nu)} = 0$, and hence the $\hat{K}^\sigma_\lambda$ in the first term of (5.8) can be replaced by K^σ_λ. The second term of (5.8) comes from varying the vielbeins and it is given by

$$t^\nu_\mu = \frac{\mu^\epsilon}{\epsilon}\frac{1}{2}\hat{K}^\alpha_\rho\hat{K}^\beta_\sigma\delta^{\rho\sigma\nu}_{\alpha\beta\mu} = \frac{\mu^\epsilon}{\epsilon}\frac{1}{2}K^\alpha_\rho K^\beta_\sigma\delta^{\rho\sigma\nu}_{\alpha\beta\mu} + \frac{1}{2}\Big(\frac{K^2}{4}h^\nu_\mu - KK^\alpha_\rho\delta^{\rho\nu}_{\alpha\mu}\Big)\,. \tag{5.9}$$

The second contribution in (5.9) is finite and obtained from contracting h^α_β in $\hat{K}^\alpha_\beta$ with the generalized delta function. For the ball-like geometry, this stress tensor (and also this type of boundary anomaly itself) vanishes. For a cylinder with a radius r_0, using the metric and extrinsic curvatures given in (4.58) and (4.59), we have

$$K^\alpha_\rho K^\beta_\sigma\delta^{\rho\sigma\nu}_{\alpha\beta\mu} = h^\nu_\mu(K^2-{\rm tr}\,K^2)-2(KK^\nu_\mu-K^\rho_\mu K^\nu_\rho) = \frac{h^\nu_\mu}{r_0^2}(d-2)(d-3)-\frac{2}{r_0^2}\bar{\delta}^\nu_\mu(d-3)\,, \tag{5.10}$$

$$\left(\frac{K^2}{4}h^\nu_\mu - KK^\alpha_\rho\delta^{\rho\nu}_{\alpha\mu}\right)|_{d\to3} = -\frac{1}{r_0^2}\left(\frac{3}{4}h^\nu_\mu-\bar{\delta}^\nu_\mu\right)\,. \tag{5.11}$$

Thus, we obtain $t^\nu_\mu = \frac{1}{2r_0^2}{\rm diag}\Big(0,\frac{1}{4},-\frac{3}{4}\Big)$. The first term in (5.8) still has the remaining $\frac{1}{\epsilon}$ factor that causes the divergence. We again add an Weyl invariant regulator

$$\widetilde{W}^{\rm reg} = \frac{\mu^\epsilon}{\epsilon}c'\int_{\partial\mathcal{M}} d^2x\sqrt{h}\ {\rm tr}\,\hat{K}^2\,, \tag{5.12}$$

with the coefficient $c' = -\frac{c_3}{256\pi}$. The (renormalized) stress tensor is given by

$$\langle T^\nu_\mu\rangle^{(c_3)}|_{\rm cylinder} = \frac{c_3}{256\pi r_0^2}{\rm diag}\Big(0,-\frac{1}{4},\frac{3}{4}\Big)\delta(r-r_0)\,. \tag{5.13}$$

Taking the trace on this stress tensor gives

$$\langle T^\mu_\mu\rangle^{(c_3)}|_{\rm cylinder} = \frac{c_3}{512\pi r_0^2}\delta(r-r_0)\,, \tag{5.14}$$

which recovers the anomaly for a $d=3$ cylinder where

$$\mathring{R}=0, \quad \mathrm{tr}\,\hat{K}^2=\frac{1}{2r_0^2}\,. \tag{5.15}$$

On the other hand, there can be a_3-type stress tensors near the boundary. We have

$$\langle T^\mu_\nu(x)\rangle^{(a_3)}=\frac{a_3}{768\pi}\frac{\mu^\epsilon}{\epsilon}\mathring{R}^{\lambda\sigma}{}_{\rho\delta}\delta^{\rho\delta\mu}_{\lambda\sigma\nu}\delta(x_\perp)=-\frac{a_3}{192\pi}\frac{\mu^\epsilon}{\epsilon}\mathring{E}^\mu_\nu\delta(x_\perp)\,. \tag{5.16}$$

($\mathring{E}^\mu_\nu$ is the boundary Einstein tensor.) For a 3-ball with the radius r_0, we obtain

$$\langle T^\mu_\nu\rangle^{(a_3)}_{\rm ball}=\frac{a_3}{192\pi r_0^2}\mathrm{diag}\Big(0,\frac{1}{2},\frac{1}{2}\Big)\delta(\mathrm{r}-\mathrm{r}_0)\,. \tag{5.17}$$

The stress tensor trace,

$$\langle T^\mu_\mu\rangle^{(a_3)}_{\rm ball}=\frac{a_3}{192\pi r_0^2}\delta(r-r_0)\,, \tag{5.18}$$

recovers the Weyl anomaly evaluated for a 3-ball where

$$\mathring{R}=\frac{2}{r_0^2},\quad \hat{K}_{\mu\nu}=0\,. \tag{5.19}$$

For a 3-cylinder with the radius r_0, we have

$$\langle T^\mu_\nu\rangle^{(a_3)}_{\rm cylinder}=\frac{a_3}{192\pi r_0^2}\mathrm{diag}\Big(0,\frac{1}{2},-\frac{1}{2}\Big)\delta(\mathrm{r}-\mathrm{r}_0)\,. \tag{5.20}$$

This result is traceless, as expected, since the a_3-type anomaly vanishes for a 3-cylinder.

6 Conclusion

The general motivation of this paper comes from the constraints of the RG flows in both even-and odd-dimensional QFTs in flat manifolds with a boundary, and also comes from the universal contribution to the entanglement entropy. It is certainly of great interest to find more physical quantities characterized by boundary terms of conformal anomalies. Here we investigate the stress tensor correlation functions in flat spacetime with a generally curved boundary, focusing on the contribution from the boundary counter-terms. In particular, in $d=4$, we find that the conformal Ward identity is modified by boundary central charges and the charge b_1 gives an additional correction to the RG equation near the boundary. Moreover, the boundary counter-terms induce new stress tensors near the boundary. We have considered examples using a ball and a cylinder. We also discussed the similar story for $d=3$ CFTs, where the Weyl anomaly exists only on the boundary. It will be interesting to generalize these results to five and six dimensions.

Let us conclude by listing some questions that relate to boundary Weyl anomalies:

(1) How do the boundary central charges modify the n-point functions of the stress tensor at non-zero separation?[9]

[9] In particular, the bulk anomalies can be measured by putting all the three stress tensors at different points. It would be interesting to clarify whether this is true for boundary anomalies and how to exactly measure them from separated points correlation functions.

(2) For four-dimensional QFTs with a boundary, (i) is it generally true that the edge central charge satisfies $b_{1(\mathrm{UV})} > b_{1(\mathrm{IR})}$? (ii) Does the bulk a-charge still satisfy monotonicity under the RG flow in the presence of a boundary? (The boundary terms of the dilaton effective action calculated in [17] might be useful.)

It will be also interesting to interpret these boundary Weyl anomalies in the AdS/CFT correspondence.

Acknowledgments

I would like to thank Chris Herzog and Kristan Jensen for reading the manuscript and for their useful comments. The work was supported in part by the NSF under Grant No. PHY13-16617.

A Metric variation

For the convenience of the reader, here we list formulae for the metric variation of curvatures.

Under the metric perturbation $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$, we have

$$g^{\mu\nu} \to g^{\mu\nu} - g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta} + g^{\mu\alpha} g^{\nu\beta} g^{\lambda\rho} \delta g_{\alpha\lambda} \delta g_{\beta\rho} + \cdots , \tag{A.1}$$

$$\sqrt{g} \to \sqrt{g} + \frac{1}{2}\sqrt{g} g^{\mu\nu} \delta g_{\mu\nu} + \cdots , \tag{A.2}$$

$$\delta^{(n)}\Gamma^{\lambda}_{\mu\nu} = \frac{n}{2}\delta^{(n-1)}(g^{\lambda\rho})\Big(\nabla_\mu \delta g_{\rho\nu} + \nabla_\nu \delta g_{\rho\mu} - \nabla_\rho \delta g_{\mu\nu}\Big) , \tag{A.3}$$

$$\delta R^{\lambda}_{\ \mu\sigma\nu} = \nabla_\sigma \delta\Gamma^{\lambda}_{\mu\nu} - \nabla_\nu \delta\Gamma^{\lambda}_{\mu\sigma} , \tag{A.4}$$

$$\delta R_{\mu\nu} = \frac{1}{2}\Big(\nabla^\lambda \nabla_\mu \delta g_{\lambda\nu} + \nabla^\lambda \nabla_\nu \delta g_{\mu\lambda} - g^{\lambda\rho}\nabla_\mu \nabla_\nu \delta g_{\lambda\rho} - \Box \delta g_{\mu\nu}\Big) , \tag{A.5}$$

$$\delta R = -R^{\mu\nu}\delta g_{\mu\nu} + \nabla^\mu\Big(\nabla^\nu \delta g_{\mu\nu} - g^{\lambda\rho}\nabla_\mu \delta g_{\lambda\rho}\Big) . \tag{A.6}$$

Defining the induced metric by $h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$, on the boundary we have

$$\delta n_\mu = \frac{1}{2} n_\mu n^\lambda n^\nu \delta g_{\lambda\nu} , \tag{A.7}$$

$$\delta K_{\mu\nu} = \delta(h^\lambda_\mu h^\rho_\nu \nabla_\lambda n_\rho) \tag{A.8}$$

$$= \frac{n^\lambda n^\rho \delta g_{\lambda\rho} K_{\mu\nu}}{2} + \delta g_{\lambda\rho} n^\rho \Big(n_\mu K^\lambda_\nu + n_\nu K^\lambda_\mu\Big) - \frac{h^\lambda_\mu h^\rho_\nu n^\alpha}{2}\Big(\nabla_\lambda \delta g_{\alpha\rho} + \nabla_\rho \delta g_{\lambda\alpha} - \nabla_\alpha \delta g_{\lambda\rho}\Big) ,$$

$$\delta K = -\frac{1}{2}K^{\mu\nu}\delta g_{\mu\nu} - \frac{1}{2}n^\rho\Big(\nabla^\lambda \delta g_{\rho\lambda} - g^{\lambda\sigma}\nabla_\rho \delta g_{\lambda\sigma}\Big) - \frac{1}{2}\mathring{\nabla}^\mu\Big(h^\lambda_\mu n^\rho \delta g_{\lambda\rho}\Big) , \tag{A.9}$$

where $\mathring{\nabla}^\mu$ denotes the covariant derivative on the boundary.

B Two-dimensional CFTs with a boundary

For a compact $d = 2$ manifold, the Weyl anomaly is given by

$$\langle T^\mu_\mu(x)\rangle = \frac{a}{2\pi}R = \frac{c}{24\pi}R . \tag{B.1}$$

The central charge notation c is more common in the literature. One has

$$\int \sqrt{g} d^2 x R = 2\pi\chi\,, \tag{B.2}$$

where χ is the Euler number. Using the expansion on the Ricci scalar around flat space, one obtains the Ward identity

$$\langle T^{\mu}_{\mu}(x) T_{\sigma\lambda}(y)\rangle = -\frac{c}{12\pi}(\partial_\sigma\partial_\lambda - \delta_{\sigma\lambda}\Box)\delta^2(x-y)\,. \tag{B.3}$$

In the presence of a boundary one has

$$\langle T^{\mu}_{\mu}(x)\rangle = \frac{c}{24\pi}\Big(R + 2K\delta(x_\perp)\Big)\,. \tag{B.4}$$

The Ward identity is modified by the boundary term and we obtain

$$\begin{aligned}\langle T^{\mu}_{\mu}(x) T_{\sigma\lambda}(y)\rangle = &-\frac{c}{12\pi}\Big((\partial_\sigma\partial_\lambda - \delta_{\sigma\lambda}\Box)\delta^2(x-y) - \delta(x_\perp) h_{\sigma\lambda}\partial_{\rm n}\delta^2(x-y)\Big)\\ &+\delta_{\sigma\lambda}\delta^2(x-y)\langle T^{\mu}_{\mu}(x)\rangle|_{\delta_{\mu\nu}} + 2\delta^2(x-y)\langle T_{\sigma\lambda}(x)\rangle|_{\delta_{\mu\nu}}\,.\end{aligned} \tag{B.5}$$

The anomaly in the flat limit is given by

$$\langle T^{\mu}_{\mu}\rangle|_{\delta_{\mu\nu}} = \frac{c}{12\pi} K\delta(x_\perp)\,. \tag{B.6}$$

We define the vacuum stress tensor of a plane to vanish. For a disk with a radius r_0, we have

$$\langle T^{\beta}_{\alpha}\rangle|_{\delta_{\mu\nu}} = \frac{c}{12\pi}\frac{\mu^\epsilon}{\epsilon} K^{\mu}_{\nu}\delta^{\nu\beta}_{\mu\alpha}\delta(r-r_0) = \frac{c}{12\pi r_0}{\rm diag}\Big(0,1\Big)\delta(r-r_0)\,. \tag{B.7}$$

(Now we are working in $d = 2+\epsilon$ dimensions.) Only the angular component is non-vanishing. The trace of this stress tensor reproduces the anomaly in the flat limit. Notice that in $d = 2$ curved space, the bulk stress tensor has to be expressed in terms of the Weyl factor $\sigma(x)$ defined via $g_{\mu\nu} = e^{2\sigma}\delta_{\mu\nu}$, because the Einstein tensor vanishes in $d = 2$. See [17] for more discussions on $d = 2$ CFTs.

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